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A CLASSIFICATION OF ω-REGULAR LANGUAGES

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Abstract. For a given ω-regular language A we establish an invariant property of the structure of finite automata which accept A.

1. Introduction

In [4], Landweber investigated the Borel complexity of sets of infinite words definable by different kinds of finite automata and presented relevant decision algorithms. His main result is a necessary and sufficient condition for an ω-regular set to be accepted by a deterministic Büchi automaton. Similar results for a topology other than the product topology were established by Takahashi and Yamasaki [8].

In this paper we define a 'Boolean' hierarchy of ω-regular sets. A position of an ω-regular set A in this hierarchy is given by a 'minimal' Boolean expression for A over the sets definable by deterministic Büchi automata. On the other hand, we define the same hierarchy in terms of deterministic finite automata, i.e., for an ω-regular set A we establish an invariant property of finite automata which accept A. Also, we present an effective test for this property. The sets definable by deterministic Büchi automata form a hierarchy class, and the decision algorithm of Landweber [4] is a particular case of our test.

The second section contains some basic definitions, background and the statement of the main result, which is proved in the third section. Some properties of the hierarchy are given in the fourth section.

2. Definitions, background and statement of the main result

Let Σ be a finite set, the input alphabet. Σ^* (Σ^ω) denotes the set of all finite (infinite) sequences on Σ. If $x = a_1 \dots a_i \dots a_n \in \Sigma^*$ ($x = a_1 \dots a_i \dots \in \Sigma^\omega$), then $x(i)$ denotes a_i. (If $i > n$, $x(i)$ is undefined in the finite case.) ε denotes the empty sequence. If $x = a_1 \dots a_n \in \Sigma^*$, then the length $|x|$ of x is $|x| = n$, $|\varepsilon| = 0$. For $x \in \Sigma^\omega$ define $|x| = \infty$. If $x \in \Sigma^*$, $y \in \Sigma^* \cup \Sigma^\omega$, then $z = xy$ is the concatenation of x and y,

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i.e., $z(i)=x(i)$ if $i \leqslant |x|$, and $z(i)=y(i-|x|)$ otherwise. For $A, A' \subseteq \Sigma^*$ define the concatenation of A and A' as $AA'=\{xy \mid x \in A, y \in A'\}$. For $A \in \Sigma^*$, $0 \leqslant i$ define by recursion A^i as follows: $A^0=\{\varepsilon\}$, $A^{i+1}=A^iA$. A^* denotes $\bigcup_{i=0}^{\infty} A^i$. We define the partial order on $\Sigma^* \cup \Sigma^\omega$ by $x<y$ if $x(i)=y(i)$ for $i=1,\ldots,|x|$. If $x \in \Sigma^* \backslash \{\varepsilon\}$, then x^ω denotes the unique sequence α of Σ^ω satisfying $\alpha(m|x|+i)=x(i)$, $m=0,1,\ldots$, $i=1,\ldots,|x|$. $P(S)$ denotes the set of all subsets of the set S. Set inclusion is indicated by $\subseteq$, proper set inclusion by $\subset$, and $c(S)$ is the cardinality of the set S. If $S' \subseteq S$, then $\bar{S}'$ denotes the complement of S' to S. For the set S, $\alpha \in S^\omega$ define $\text{In}(\alpha)=\{s \in S \mid c(\alpha^{-1}(s))=\omega\}$, i.e., $\text{In}(\alpha)$ is the set of elements of S which appear infinitely often in α.

2.1. Definition. A *Σ-table of transitions* is a system $\mu=\langle S, s_0, \delta\rangle$, where S is a finite set of states, $s_0 \in S$ is the initial state, and δ is a mapping $\delta: S \times \Sigma \to S$. We extend δ to a mapping $\delta: S \times \Sigma^* \to S$ by $\delta(s, \varepsilon)=s$, and $\delta(s, xa)=\delta(\delta(s,x),a)$. The state $s \in S$ is called *accessible* from the state $t \in S$ if, for some $x \in \Sigma^*$, $\delta(t,x)=s$. In this case we shall say that s is accessible from t by x. If s is accessible from t, then it is accessible by some $x \in \Sigma^*$ satisfying $|x|<c(S)$. Therefore, the set of all states acceptable from t can be effectively constructed. To simplify the proofs *we always assume that all states of S are accessible from s_0.*

2.2. Definition. Let $x \in \Sigma^\omega$, and let $\mu=\langle S, s_0, \delta\rangle$ be a Σ-table of transitions. We call $r \in S^\omega$ the *run* of μ on x, denoted $r=\mu(x)$ if (i) $r(1)=s_0$, and (ii) $r(i)=\delta(r(i-1), x(i-1))$ for $i>1$.

2.3. Definition. A *deterministic Büchi automaton* (over Σ) is a system $\boldsymbol{A}=\langle S, s_0, \delta, F\rangle$, where $\mu=\langle S, s_0, \delta\rangle$ is a Σ-table of transitions, $F \subseteq S$. For $x \in \Sigma^\omega$ we say that $\boldsymbol{A}$ *accepts* x if $\text{In}(\mu(x)) \cap F \neq \emptyset$. $T(\boldsymbol{A})$ denotes the set of all elements of Σ^ω which are acceptable by $\boldsymbol{A}$: $T(\boldsymbol{A})=\{x \in \Sigma^\omega \mid \text{In}(\mu(x)) \cap F \neq \emptyset\}$. We denote the set of all sets acceptable by deterministic Büchi automata by $\boldsymbol{B}$.

2.4. Definition. A *deterministic automaton* (over Σ) is a system $\boldsymbol{A}=\langle S, s_0, \delta, \boldsymbol{F}\rangle$, where $\mu=\langle S, s_0, \delta\rangle$ is a Σ-table of transitions, and $\boldsymbol{F} \subseteq P(S)$. For $x \in \Sigma^\omega$ we say that $\boldsymbol{A}$ accepts x if $\text{In}(\mu(x)) \in \boldsymbol{F}$. $T(\boldsymbol{A})$ denotes the set of all elements of Σ^ω which are acceptable by $\boldsymbol{A}$. A subset of Σ^ω is called is called *regular* if it is acceptable by some deterministic automaton. We denote the set of all sets acceptable by deterministic automata by $\boldsymbol{D}$. (The automata described above are called 2- and 3-automata respectively, in [4] and [8], but we reserve this notation for another kind of automata.)

In the sequel by the run of an automaton we shall mean the run of its table. To proceed we need the following lemmas and definitions.

2.5. Lemma. *$\boldsymbol{B}$ is closed under union and under intersection.*

Proof. Our proof is similar to the proof of the corresponding theorem of [7]: Let $\boldsymbol{A}=\langle S, s_0, \delta, F\rangle$ and $\boldsymbol{B}=\langle S', s_0', \delta', F'\rangle$ be deterministic Büchi automata. Define

$$\boldsymbol{C}=\langle S\times S', (s_0, s_0'), \delta_C, S\times F'\cup F\times S'\rangle,$$

by $\delta_C((s, s'), a)=(\delta(s, a), \delta'(s', a))$, for $(s, s')\in S\times S', a\in\Sigma$. Clearly, $T(\boldsymbol{C})=T(\boldsymbol{A})\cup T(\boldsymbol{B})$. Define

$$\boldsymbol{D}=\langle S+S'\times\{0,1,2\}, (s_0, s_0', 0), \delta_D, S\times S'\times\{2\}\rangle$$

by $\delta_D((s, s', i), a)=(\delta(s, a), \delta'(s', a), j)$, where $j=1$ if and only if $i=0$, and $s\in F$. or $i=1$, and $s'\notin F'$; $j=2$ if and only if $i=1$, and $s'\in F'$, $j=0$ if and only if $i=2$, or $i=0$, and $s\notin F$. We have $T(\boldsymbol{D})=T(\boldsymbol{A})\cap T(\boldsymbol{B})$. $\square$

Another (topological) proof can be easily obtained from [4].

2.6. Corollary. *Let* **CB** *denote the Boolean closure of* $\boldsymbol{B}$, $A\in \mathbf{CB}$. *There exist* $B_1, B_1', \ldots, B_n, B_n'\in \boldsymbol{B}$ *such that* $A=\bigcup_{i=1}^{n}(B_i\cap \bar{B}_i')$.

Proof. Let $A=F(C_1, \ldots, C_m)$, where $C_1, \ldots, C_m\in \boldsymbol{B}$, F is a Boolean function. Put F into the normal $\cup$-$\cap$ form:

$$F(C_1, \ldots, C_m)=\bigcup_k\left(\bigcap_{i\in I_k} C_i\cap \bar{C}_J\right)=\bigcup_k\left(\bigcap_{i\in I_k} C_i\cap \overline{\bigcup_{j\in J_k} C_j}\right).$$

By Lemma 2.5, $B_k=\bigcap_{i\in I_k} C_i$, $B_k'=\bigcup_{j\in J_k} C_j\in \boldsymbol{B}$. $\square$

It is known from [7] that **CB** coincides with $\boldsymbol{D}$. In the sequel we shall give a proof of this result (different from the proof of [7]), which will serve our purposes. Now we define a Boolean hierarchy of ω-regular sets.

2.7. Definition. Define the sequence $\boldsymbol{B}_1\subseteq \boldsymbol{B}_2\subseteq\cdots\subseteq \boldsymbol{B}_n\subseteq\cdots$ as follows:

$$\boldsymbol{B}_n=\left\{A\in \mathbf{CB}\mid \text{there exist } A_1, A_1', \ldots, A_n, A_n'\in \boldsymbol{B}, \text{ such that } A=\bigcup_{i=1}^{n}(A_i\cap \overline{A_i'})\right\}.$$

It follows from Corollary 2.6 that $\bigcup_{n=1}^{\infty}\boldsymbol{B}_n=\mathbf{CB}$.

Further define the sets $\mathbf{LB}_n$, $\mathbf{RB}_n$ and $\mathbf{LRB}_n$ as follows:

$$\mathbf{LB}_n=\left\{A\in \mathbf{CB}\mid \text{ there exist } A_1', A_2, A_2', \ldots, A_n, A_n'\in \boldsymbol{B} \text{ such that } A=A_1'\cup\bigcup_{i=2}^{n}(A_i\cap \bar{A}_i')\right\}.$$

$$\mathbf{RB}_n = \Big\{ A \in \mathbf{CB} \mid \text{there exist } A_1, A_1', \ldots, A_{n-1}, A_{n-1}', A_n \in \boldsymbol{B}, \text{ such that}$$

$$A = \bigcup_{i=1}^{n-1} (A_i \cap \bar{A}_i') \cup A_n \Big\}.$$

$$\mathbf{LRB}_n = \Big\{ A \in \mathbf{CB} \mid \text{there exist } A_1, A_2, A_2', \ldots, A_{n-1}, A_{n-1}', A_n \in \boldsymbol{B}, \text{ such that}$$

$$A = \bar{A}_1' \cup \bigcup_{i=2}^{n-1} (A_i \cap \bar{A}_i') \cup A_n \Big\} \quad (\text{for } n > 1).$$

Note that $\mathbf{RB}_1 = \boldsymbol{B}$. Obviously:

$$\begin{tikzcd}
\mathbf{LB}_{n-1} \subseteq \mathbf{LRB}_n & & \mathbf{LB}_n \\
& \mathbf{LB}_n \cap \mathbf{RB}_n \arrow[ul, phantom, sloped, "\subseteq"] \arrow[ur, phantom, sloped, "\subseteq"] & \\
\mathbf{RB}_{n-1} \subseteq \boldsymbol{B}_{n-1} \arrow[ur, phantom, sloped, "\subseteq"] & & \mathbf{RB}_n \subseteq \boldsymbol{B}_n. \arrow[ul, phantom, sloped, "\supseteq"]
\end{tikzcd}$$

In Section 3 we give an equivalent definition of the hierarchy above. We prove that all inclusions are proper, and that $\mathbf{LRB}_n/\boldsymbol{B}_{n-1} \neq \emptyset$ and $\boldsymbol{B}_{n-1}/\mathbf{LRB}_n \neq \emptyset$. For an ω-regular set A we establish an invariant property of deterministic automata which accept A. This property is determined by the position of A in this hierarchy. There is, also, a connection between the position of A and the minimal size of a boolean formula which defines A over $\boldsymbol{B}$ (see Section 4).

2.8. Definition. Let $\mu = \langle S, s_0, \delta \rangle$ be a Σ-table of transitions (recall that all states of μ are accessible from s_0). $S' \subseteq S$ is called a *realizable cycle* of μ if, for some $s \in S$, there is some $x \in \Sigma^*$, such that $s = \delta(s, x)$, $S' = \{\delta(s, x(1) \ldots x(i)) \mid 1 \leqslant i \leqslant |x|\}$. In this case we say that x is an s-realization of S', and S' is the cycle realizable by x. Let $R(\mu)$ denote the set of all realizable cycles of μ. By the *set of realizable cycles* of an automaton we mean the set of realizable cycles of its transition table.

Obviously, if $\boldsymbol{A} = \langle S, s_0, \delta, \boldsymbol{F} \rangle$ is a deterministic automaton $\boldsymbol{B} = \langle S, s_0, \delta, \boldsymbol{F} \cap R(\boldsymbol{A}) \rangle$, then $T(\boldsymbol{A}) = T(\boldsymbol{B})$.

It is easy to prove the following proposition.

2.9. Proposition. *For any Σ-table of transitions μ, $R(\mu)$ can be effectively constructed.*

2.10. Definition. Let $\boldsymbol{F} \subseteq \boldsymbol{G} \subseteq P(S)$. $\boldsymbol{F}$ is *dense* (in $\boldsymbol{G}$) if, for every $S_1, S_2 \in \boldsymbol{F}$, $S_3 \in \boldsymbol{G}$, $S_1 \subseteq S_3 \subseteq S_2$ implies that $S_3 \in \boldsymbol{F}$. $\boldsymbol{F}$ is R-dense if, for every $S_1 \in \boldsymbol{F}, S_2 \in \boldsymbol{G}$, $S_1 \subseteq S_2$ implies that $S_2 \in \boldsymbol{F}$. $\boldsymbol{F}$ is L-dense if, for every $S_1 \in \boldsymbol{F}$, $S_2 \in \boldsymbol{G}$, $S_2 \subseteq S_1$ implies that $S_2 \in \boldsymbol{F}$.

Let $\boldsymbol{F} = \boldsymbol{F}_1 \cup \boldsymbol{F}_2 \cup \cdots \cup \boldsymbol{F}_n$. $\{\boldsymbol{F}_1, \ldots, \boldsymbol{F}_n\}$ is called an *n-decomposition* of $\boldsymbol{F}$ (in $\boldsymbol{G}$) if every $\boldsymbol{F}_i$ is dense, $i = 1, \ldots, n$. An n-decomposition of $\boldsymbol{F}$, $\{\boldsymbol{F}_1, \ldots, \boldsymbol{F}_n\}$, is called an *$(L-n)$-decomposition*, if $\boldsymbol{F}_1$ is L-dense. It is called an *$(R-n)$-decomposition*, if

F_n is R-dense. It is called an *$(LR-n)$-decomposition*, if $n>1$, F_1 is L-dense, and F_n is R-dense.

We can classify deterministic automata as follows.

2.11. Definition. $\boldsymbol{D}_n = \{A \mid A = \langle S, s_0, \delta, F\rangle$ is a deterministic automaton, and there exists an n-decomposition of F in $R(A)\}$. $\mathbf{LD}_n = \{A \mid A = \langle S_1, s_0, \delta, F\rangle$ is a deterministic automaton, and there exists an $(L-n)$-decomposition of F in $R(A)\}$. $\mathbf{RD}_n = \{A \mid A = \langle S, s_0, \delta, F\rangle$ is a deterministic automaton, and there exists an $(R-n)$-decomposition of F in $R(A)\}$. $\mathbf{LRD}_n = \{A \mid A = \langle S, s_0, \delta, F\rangle$ is a deterministic automaton, and there exists an $(LR-n)$-decomposition of F in $R(A)\}$ (for $n>1$).

Our main result is the following.

2.12. Theorem
(a) *$A \in \boldsymbol{D}_n$ if and only if $T(A) \in \boldsymbol{B}_n$.*
(b) *$A \in \mathbf{LD}_n$ if and only if $T(A) \in \mathbf{LB}_n$.*
(c) *$A \in \mathbf{RD}_n$ if and only if $T(A) \in \mathbf{RB}_n$.*
(d) *$A \in \mathbf{LRD}_n$ if and only if $T(A) \in \mathbf{LRB}_n$.*

3. Proof of Theorem 2.12

We need an equivalent description of the hierarchy defined in Section 2. For this we present another version of deterministic automata introduced in [2] and [7].

3.1. Definition. For $n \geq 1$ define an *n-automaton* as a system $A = \langle S_1, s_0, \delta, \{\langle L_1, R_1\rangle, \ldots, \langle L_n, R_n\rangle\}\rangle$, where $\mu = \langle S, s_0, \delta\rangle$ is a Σ-table of transitions, and $L_1, R_1, \ldots, L_n, R_n \subseteq S$. $x \in \Sigma^\omega$ is acceptable by A if, for some $1 \leq i \leq n$, $\mathrm{In}(\mu(x)) \cap L_i = \emptyset$, and $\mathrm{In}(\mu(x)) \cap R_i \neq \emptyset$. As above, by the *run of an n-automaton* we mean the run of its table of transitions. $T(A)$ denotes the set of all infinite words acceptable by A: $T(A) = \{x \in \Sigma^\omega \mid$ for some $1 \leq i \leq n$, $\mathrm{In}(A(x)) \cap L_i = \emptyset$, and $\mathrm{In}(A(x)) \cap R_i \neq \emptyset\}$.

A is called an $(L-n)$-automaton if $R_1 = S$. A is called an $(R-n)$-automaton if $L_n = \emptyset$. A is called an $(LR-n)$-automaton $(n \geq 2)$ if $L_n = \emptyset$, and $R_1 = S$.

Note. $(R-1)$-automata are exactly deterministic Büchi automata.

$\boldsymbol{M}_n$ denotes the set of all n-automata. $\mathbf{LM}_n$ denotes the set of all $(L-n)$-automata. $\mathbf{RM}_n$ denotes the set of all $(R-n)$-automata. $\mathbf{LRM}_n$ denotes the set of all $(LR-n)$-automata.

3.2. Lemma
(a) *If $A \in \boldsymbol{M}_n$, then $T(A) \in \boldsymbol{B}_n$.*
(b) *If $A \in \mathbf{LM}_n$, then $T(A) \in \mathbf{LB}_n$.*

(c) *If* $A \in \mathbf{RM}_n$, *then* $T(A) \in \mathbf{RB}_n$.
(d) *If* $A \in \mathbf{LRM}_n$, *then* $T(A) \in \mathbf{LRB}_n$.

Proof. Let $A = \langle S, s_0, \delta, \{\langle L_1, R_1\rangle, \ldots, \langle L_n, R_n\rangle\}\rangle$ be an n-automaton. Define deterministic Büchi automata $A_1, \ldots, A_n, A', \ldots, A'_n$ as follows:

$$A_i = \langle S, s_0, \delta, R_i\rangle, \quad A'_i = \langle S, s_0, \delta, L_i\rangle, \qquad i = 1, \ldots, n.$$

Obviously,

$$T(A) = \bigcup_{i=1}^{n} (T(A_p) \cap \overline{T(A'_i)}).$$

Moreover, if $A \in \mathbf{LM}_n (R_1 = S)$, $T(A_1) = \Sigma^*$, $T(A) \cap \overline{T(A'_1)} = T(A'_1)$, and $T(A) \in \mathbf{LB}_n$. The proofs of points (c) and (d) are similar to the above. □

3.3. Lemma

(a) *If* $A \in \boldsymbol{B}_n$, *then, for some* $A \in \boldsymbol{M}_n$, $A = T(A)$.
(b) *If* $A \in \mathbf{LB}_n$, *then, for some* $A \in \mathbf{LM}_n$, $A = T(A)$.
(c) *If* $A \in \mathbf{RB}_n$, *then, for some* $A \in \mathbf{RM}_n$, $A = T(A)$.
(d) *If* $A \in \mathbf{LRB}_n$, *then, for some* $A \in \mathbf{LRM}_n$, $A = T(A)$ $(n \geq 2)$.

Proof. We prove the lemma by induction on n.

Basis. $n = 1$. $A = T(A_1) \cap \overline{T(A'_1)}$, where $A_1 = \langle S, s_0, \delta, F\rangle$, $A'_1 = \langle S', s'_0, \delta', F'\rangle$ are deterministic Büchi automata. Define a 1-automaton $A = \langle S \times S_1, \{s_0, s'_0\}, \delta_A, \{\langle L_1, R_1\rangle\}\rangle$ by

$$\delta_A((s, s'), a) = (\delta(s, a), \delta'(s', a)), \quad L_1 = S \times F', \ R_1 = F \times S'.$$

Obviously, $T(A) = T(A_1) \cap \overline{T(A'_1)} = A$. If $A = T(A_1) \in \mathbf{RB}_1$, let $A = \langle S, s_0, \delta, \{\langle \emptyset, F\rangle\}\rangle$; clearly, $A = T(A)$. If $A = \overline{T(A'_1)} \in \mathbf{LB}_1$, let $A = \langle S', s'_0, \delta', \{\langle F', S'\rangle\}\rangle$; clearly $A = T(A)$.

Inductive step. Let $A \in \boldsymbol{B}_{n+1}$; then, for some $A' \in \boldsymbol{B}_n$, $A'' \in \boldsymbol{B}_1$, $A = A' \cup A''$.

By the inductive hypothesis there exist an n-automaton $A' = \langle S, s'_0, \delta, \{\langle L_1, R_1\rangle, \ldots, \langle L_n, R_n\rangle\}\rangle$ and a 1-automaton $A'' = \langle S'', s'_0, \delta', \{\langle L_{n+1}, R_{n+1}\rangle\}\rangle$ such that $A' = T(A')$ and $A'' = T(A'')$.

Define an $(n+1)$-automaton

$$A = \langle S \times S', (s_0, s'_0), \delta_A, \{\langle L_1^A, R_1^A\rangle, \ldots, \langle L_{n+1}^A, R_{n+1}^A\rangle\}\rangle$$

by

$$\delta_A((s, s'), a) = (\delta(s, a), \delta(s', a)),$$

$$L_i^A = L_i \times S', \ R_i^A = R_i \times S' \quad \text{for } i = 1, \ldots, n$$

$$L_{n+1}^A = S \times L_{n+1}, \ R_i^A = S \times R_{n+1}.$$

It is easy to prove that $T(A) = A$. (See [7] for details.)

To prove point (b) of the lemma, observe that, if $A \in \mathbf{LB}_{n+1}$, we may suppose, by the inductive hypothesis, that $A' \in \mathbf{LB}_n$. Therefore, $A' \in \mathbf{LM}_n$, $R_1 = S$, and $R_1^A = S \times S_1$. Points (c) and (d) can be proved in a similar fashion. □

3.4. Lemma

(a) *If* $A \in \boldsymbol{B}_n$, *then, for some* $\boldsymbol{A} \in \boldsymbol{D}_n$, $A = T(\boldsymbol{A})$.
(b) *If* $A \in \mathbf{LB}_n$, *then, for some* $\boldsymbol{A} \in \mathbf{LD}_n$, $A = T(\boldsymbol{A})$.
(c) *If* $A \in \mathbf{RB}_n$, *then, for some* $\boldsymbol{A} \in \mathbf{RD}_n$, $A = T(\boldsymbol{A})$.
(d) *If* $A \in \mathbf{LRB}_n$, *then, for some* $\boldsymbol{A} \in \mathbf{LRD}_n$, $A = T(\boldsymbol{A})$.

Proof. Let $A \in \boldsymbol{B}_n$, then, by Lemma 3.3 there exists an n-automaton $\boldsymbol{A}' = \langle S, s_0, \delta, \{\langle L, R, \rangle, \ldots, \langle L_n, R_n \rangle\}\rangle$ such that $A = T(\boldsymbol{A}')$. Let $\boldsymbol{F}_i = \{F \in R(\boldsymbol{A}') \mid F \cap L_i = \emptyset$, and $F \cap R_i \neq \emptyset\}$, $i = 1, \ldots, n$, $\boldsymbol{F} = \bigcup_{i=1}^n \boldsymbol{F}_i$. For a deterministic automaton $\boldsymbol{A} = \langle S, s_0, \delta, \boldsymbol{F}\rangle$, $A = T(\boldsymbol{A})$: for any $x \in \Sigma^\omega$, $\boldsymbol{A}(x) = \boldsymbol{A}'(x)$, because $\boldsymbol{A}$ and $\boldsymbol{A}'$ have the same table of transitions, and $\mathrm{In}(\boldsymbol{A}(x)) \in \boldsymbol{F}_i$ if and only if $\mathrm{In}(\boldsymbol{A}'(x)) \cap L_i = \emptyset$, and $\mathrm{In}(\boldsymbol{A}'(x)) \cap R_i \neq \emptyset$. Every $\boldsymbol{F}_i$ $(i = 1, \ldots, n)$ is dense in $R(\boldsymbol{A})$: for $S_1, S_2 \in \boldsymbol{F}_i$, $S_3 \in R(\boldsymbol{A})$ such that $S_1 \subseteq S_3 \subseteq S_2$, $S_3 \cap L_i \subseteq S_2 \cap L_i = \emptyset$, and $\emptyset \neq S_1 \cap R_i, \subseteq S_3 \cap R_i$. Therefore, $S_3 \in \boldsymbol{F}_i$. It follows that $\boldsymbol{A} \in \boldsymbol{D}_n$. If $A \in \mathbf{LB}_n$, we can suppose that $\boldsymbol{A}' \in \mathbf{LM}_n$, i.e., $R_1 = S$. We claim that $\boldsymbol{F}_1$ is L-dense in $R(\langle S, s_0, \delta\rangle\}$:

$$\boldsymbol{F}_1 = \{F \in R(\boldsymbol{A}') \mid F \cap L_1 = \emptyset, \text{and } F \cap R_1 \neq \emptyset\} = \{F \in R(\boldsymbol{A}') \mid F \cap L_1 = \emptyset\}.$$

Now, if $S_1 \in \boldsymbol{F}_1$, $S_2 \in R(\boldsymbol{A}')$, $S_2 \subseteq S$, then $S_2 \cap L_1 \subseteq S_1 \cap L_1 = \emptyset$. Points (c) and (d) can be proved in a similar fashion. □

3.5. Lemma ([4]). *If* $\boldsymbol{A} = \langle S, s_0, \delta, \boldsymbol{F}\rangle \in \mathbf{RD}_1$, *then* $T(\boldsymbol{A}) \in \boldsymbol{B}$.

Proof. Enumerate the elements of $\boldsymbol{F}$ by $\boldsymbol{F} = \{F_1, \ldots, F_k\}$. Define a deterministic Büchi automaton $\boldsymbol{B} = \langle \bar{S}, \overline{s_0}, \bar{\delta}, F\rangle$ by

$$\bar{S} = P(F_1) \times \cdots \times P(F_k) \times S,$$

$$\overline{s_0} = (\emptyset, \ldots, \emptyset, s_0),$$

$$\bar{\delta}((S_1, \ldots, S_i, \ldots, S_k, s), a) = (S'_1, \ldots, S'_i, \ldots, S'_k, \delta(s, a)), \text{ where}$$

$$S'_i = \emptyset \text{ if } S_i = F_i, \text{ and } S'_i = S_i \cap (F_i \cup \delta(s, a)) \text{ if } S_i \neq F_i \quad \text{for } i = 1, \ldots, k,$$

$$F = \{(S_1, \ldots, S_i, \ldots, S_k, s) \in \bar{S} \mid \text{for some } i, S_i = F_i\}.$$

Now if $x \in T(\boldsymbol{A})$, then, for some $1 \leqslant i \leqslant k$, $\mathrm{In}(\boldsymbol{A}(x)) = F_i$. Hence in $\boldsymbol{B}(x)$ the ith coordinate is equal to F_i infinitely often, and, therefore, $x \in T(\boldsymbol{B})$. We have proved that $T(\boldsymbol{A}) \subseteq T(\boldsymbol{B})$.

To see that $T(\boldsymbol{B}) \subseteq T(\boldsymbol{A})$, we note the following: If $x \in T(\boldsymbol{B})$, then for some $1 \leqslant i \leqslant k$, the ith coordinate of $\boldsymbol{B}(x)$ is equal to F_i infinitely often. Therefore, by the definition of $\bar{S}$, $F_i \subseteq \mathrm{In}(\boldsymbol{A}(x)) = F' \in R(\boldsymbol{A})$. Since $\boldsymbol{F}$ is R-dense in $R(\boldsymbol{A})$, $F' \in \boldsymbol{F}$. Thus $x \in T(\boldsymbol{A})$. This immediately yields the desired result. □

3.6. Lemma

(a) *If* $A \in D_n$, *then* $T(A) \in B_n$.

(b) *If* $A \in \mathbf{LD}_n$, *then* $T(A) \in \mathbf{LB}_n$.

(c) *If* $A \in \mathbf{RD}_n$, *then* $T(A) \in \mathbf{RB}_n$.

(d) *If* $A \in \mathbf{LRD}_n$, *then* $T(A) \in \mathbf{LRB}_n$.

Proof. Let $A = \langle S, s_0, \delta, F\rangle \in D_n$, $F_1, \ldots, F_n$ be an n-decomposition of F in $R(A)$.Let $B_i = \langle S, s_0, \delta, F_i\rangle$, $i = 1, \ldots, n$. Obviously, $T(A) = \bigcup_{i=1}^n T(B_i)$. We claim that $T(B_i) \in B_1$ $(i = 1, \ldots, n)$. Given i, define two sets F_i' and F_i'', and three deterministic automata A_i, A_i'' and A_i' as follows:

$$F_i' = \{F \in R(A)\ (= R(B_i))\,|\,\text{for some } F' \in F_i,\ F' \subseteq F\},$$

$$F_i'' = \{F \in R(A)\,|\,\text{for some } F' \in F_i,\ F \subseteq F'\},$$

$$A_i = \langle S, s_0, \delta, F_i'\rangle, \qquad A_i'' = \langle S, s_0, \delta, F_i''\rangle,$$

$$A_i' = \langle S, s_0, \delta, R(A)/F_i''\rangle.$$

Obviously, $\overline{T(A_i')} = T(A_i'')$. Since F_i is dense in $R(A) = R(B_i)$, $F_i = F_i' \cap F_i''$. Therefore,

$$T(B_i) = T(A_i) \cap T(A_i'') = T(A_i) \cap \overline{T(A_i')}.$$

F_i' is R-dense in $R(A_i) = R(A)$, and F_i'' is L-dense in $R(A_i)$. This fact follows from the definition of F_i' and F_i''. It is an easy exercise to show that $R(A)\backslash F_i''$ is R-dense in $R(A_i')$ $(= R(A))$. By Lemma 3.5, $T(A_i)$, $T(A_i') \in B$. If $A \in \mathbf{LD}_n$, then F_1 is L-dense in $R(A) = R(B_1)$. Thus, by Lemma 3.5, $T(B_i) = T(\langle S, s_0, \delta, R(A)\backslash F_1\rangle) \in \mathbf{LB}_1$. One can give similar proofs for points (c) and (d) of the lemma. □

Part 'only if' of Theorem 2.12 follows from Lemma 3.6. Part 'if' of Theorem 2.12 follows from Lemma 3.4 and the 'hierarchy' lemma, stated below.

3.7. Lemma. *Let* $A = \langle S, s_0, \delta, F\rangle$, $A' = \langle S', s_0', \delta', F'\rangle$ *be deterministic automata,* $T(A') = T(A)$.

(a) $A \in D_n$ *if and only if* $A' \in D_n$.

(b) $A \in \mathbf{LD}_n$ *if and only if* $A' \in \mathbf{LD}_n$.

(c) $A \in \mathbf{RD}_n$ *if and only if* $A' \in \mathbf{RD}_n$.

(d) $A \in \mathbf{LRD}_n$ *if and only if* $A' \in \mathbf{LRD}_n$.

To prove Lemma 3.7 we need some auxiliary lemmas and definitions.

3.8. Lemma. *Let* $F \subseteq G \subseteq P(S)$.

(a) F *does not have an n-decomposition in* G *if and only if there exist* $F_1, \ldots, F_{n+1} \in F$, $D_1, \ldots, D_n \in G/F$, *such that* $F_1 \subset D_1 \subset F_2 \subset \cdots \subset D_n \subset F_{n+1}$.

(b) F *does not have an* $(L-n)$*-decomposition in* G *if and only if there exist* $F_1, \ldots, F_n \in F$, $D_1, \ldots, D_n \in G/F$, *such that* $D_1 \subset F_1 \subset \cdots \subset D_n \subset F_n$.

(c) *$\boldsymbol{F}$ does not have an $(R-n)$-decomposition in $\boldsymbol{G}$ if and only if there exist $F_1, \ldots, F_n \in \boldsymbol{F}$, $D_1, \ldots, D_n \in \boldsymbol{G}/\boldsymbol{F}$, such that $F_1 \subset D_1 \subset \cdots \subset F_n \subset D_n$.*

(d) *$\boldsymbol{F}$ does not have an $(LR-n)$-decomposition in $\boldsymbol{G}$ if and only if there exist $F_1, \ldots, F_{n-1} \in f$, $D_1, \ldots, D_n \in \boldsymbol{G}/\boldsymbol{F}$, such that $D_1 \subset F_1 \subset \cdots \subset F_{n-1} \subset D_n$.*

Proof. We give a proof of point (a). Proofs of points (b), (c), and (d) are similar.

'*If*'. Let $F_1, \ldots, F_{n+1} \in \boldsymbol{F}$, $D_1, \ldots, D_n \in \boldsymbol{G}/\boldsymbol{F}$; $F_1 \subset D_1 \subset \cdots \subset D_n \subset F_{n+1}$. Suppose that there exists an n-decomposition of $\boldsymbol{F}$ in $\boldsymbol{G}-\{\boldsymbol{F}_1, \ldots, \boldsymbol{F}_n\}$. Then, for some $1 \leq k \leq n$, $1 \leq i < j \leq n+1$, $F_i, F_j \in \boldsymbol{F}_k$. Since $\boldsymbol{F}_k$ is dense in $\boldsymbol{G}$, $F_i \subset D_i \subset F_j$, $D_i \in \boldsymbol{F}_k$ $(\subseteq \boldsymbol{F})$. But $D_i \in \boldsymbol{G}/\boldsymbol{F}$. This contradicts our assumption. Hence $\boldsymbol{F}$ has no n-decomposition in $\boldsymbol{G}$.

'*Only if*'. Suppose that there are no $F_1, \ldots, F_{n+1} \in \boldsymbol{F}$; $D_1, \ldots, D_n \in \boldsymbol{G}/\boldsymbol{F}$, such that $F_1 \subset D_1 \subset \cdots \subset D_n \subset F_{n+1}$. To construct an n-decomposition of $\boldsymbol{F}$ in $\boldsymbol{G}$ we define subsets of $\boldsymbol{F}$, $\boldsymbol{F}_1, \ldots, \boldsymbol{F}_n$ as follows: $\boldsymbol{F}_1 = \{F \in \boldsymbol{F} \mid \text{there is no } F' \in \boldsymbol{F}, D \in \boldsymbol{G}/\boldsymbol{F}, \text{ such that } F' \subset D \subset F\}$.

If $\boldsymbol{F}_i$ has been defined, we define $\boldsymbol{F}_{i+1}$ as $\boldsymbol{F}_{i+1} = \{F \in \boldsymbol{F}/\bigcup_{k=1}^{i} \boldsymbol{F}_k \mid \text{there is no } F' \in \boldsymbol{F}/\bigcup_{k=1}^{i} \boldsymbol{F}_k, D \in \boldsymbol{G}/\boldsymbol{F}, \text{ such that } F' \subset D \subset F\}$ $(1 < i < n)$. Obviously, each $\boldsymbol{F}_i$ $(i = 1, \ldots, n)$ is dense in $\boldsymbol{G}$. Since there are no $F_1, \ldots, F_{n+1} \in \boldsymbol{F}$, $D_1, \ldots, D_n \in \boldsymbol{G}/\boldsymbol{F}$ such that $F_1 \subset D_1 \subset \cdots \subset D_n \subset F_n$, $\boldsymbol{F} = \bigcup_{i=1}^{n} \boldsymbol{F}_i$. Hence, $\{\boldsymbol{F}_1, \ldots, \boldsymbol{F}_n\}$ in an n-decomposition of $\boldsymbol{F}$ in $\boldsymbol{G}$. □

3.9. Corollary. *Let $\boldsymbol{F} \subseteq \boldsymbol{G} \subseteq P(S)$.*

(a) *$\boldsymbol{F}$ has an n-decomposition in $\boldsymbol{G}$ if and only if $\boldsymbol{G}/\boldsymbol{F}$ has an $(LR\text{-}n+1)$-decomposition in $\boldsymbol{G}$.*

(b) *$\boldsymbol{F}$ has an $(L-n)$-decomposition in $\boldsymbol{G}$ if and only if $\boldsymbol{G}/\boldsymbol{F}$ has an $(R-n)$-decomposition in $\boldsymbol{G}$.*

Proof. Point (a) follows from points (a) and (d) of Lemma 3.8. Point (b) follows from points (b) and (c) of Lemma 3.8. □

3.10. Definition. Let $\mu = \langle S, s_0, \delta \rangle$ be a Σ-table of transitions. We extend δ to a mapping $\delta: S \times P(\Sigma^*) \to P(S)$ by $\delta(s, T) = \{\delta(s, x) \mid x \in T\}$. $\delta(s, T)$ is the set of states accessible from s by the elements of T.

Let $S_1 \subseteq S$, $T \subseteq \Sigma^*$. S_1 is called *T-minimal* if, for any $s \in S_1$, $\delta(s, T) = S_1$.

3.11. Proposition. *Let $\langle S, s_0, \delta \rangle$ be a Σ-table of transitions, $\emptyset \neq T \subseteq \Sigma^*$. Let T be closed under concatenation.*

(a) *If $s' \in \delta(s, T)$, then $\delta(s', T) \subseteq \delta(s, T)$.*

(b) *For any $s \in S$ there exists a $t \in \delta(s, T)$ such that $\delta(t, T)$ is T-minimal, and $t \in \delta(t, T)$.*

Proof. (a) Let $x \in T$, $S' = \delta(s, x)$. If $r \in \delta(S', T)$, then, for some $y \in T$, $r = \delta(s', y)$. Therefore, $r = \delta(\delta(s, x), y)) = \delta(s, xy) \in \delta(s, T)$, since $xy \in T$.

(b) Let $t' \in \delta(s, T)$ be such that $\delta(t', T)$ contains the minimal number of elements. It follows from part (a) that $\delta(t', T)$ is T-minimal. Let $t \in \delta(t', T)$. Then $\delta(t, T) = \delta(t', T)$ and $t \in \delta(t, T)$. □

Proof of Lemma 3.7. By Corollary 3.9 it is sufficient to prove points (a) and (b). We prove only point (a). For this we show that $\boldsymbol{A} \notin \boldsymbol{D}_n$ if and only if $\boldsymbol{A}' \notin \boldsymbol{D}_n$. Suppose that $\boldsymbol{A} \notin \boldsymbol{D}_n$. Then, by Lemma 3.8, there exist $F_1, \ldots, F_{n+1} \in \boldsymbol{F}$, $D_1, \ldots, D_n \in R(\boldsymbol{A})/\boldsymbol{F}$, such that $F_1 \subset D_1 \subset \cdots \subset D_n \subset F_{n+1}$. We shall establish the existence of $F'_1, \ldots, F'_{n+1} \in \boldsymbol{F}'$, $D'_1, \ldots, D'_n \in R(\boldsymbol{A}')/\boldsymbol{F}'$, such that $F'_1 \subset D'_1 \subset \cdots \subset D'_n \subset F'_{n+1}$.

Let

$$N_k = \begin{cases} F_i, & k = 2i-1 \ \ (i = 1, \ldots, n+1), \\ D_j, & k = 2j \ \ (j = 1, \ldots, n). \end{cases}$$

We have $N_1 \subset N_2 \subset \cdots \subset N_{2n} \subset N_{2n+1}$.

Let $s \in N_1$. Let y_i be an s-realization of N_i $(i = 1, \ldots, 2n+1)$. For $i = 1, \ldots, 2n+1$ define $T_i = \{y_1, \ldots, y_i\}^*$.

For each $i = 1, \ldots, 2n+1$, T_i is closed under the concatenation and $T_1 \subset T_2 \subset \cdots \subset T_{2n+1}$.

Since all elements of S are accessible from s_0, for some $x \in \Sigma^*$, $s = \delta(s_0, x)$. Let $s' = \delta'(s'_0, x)$ $(\in S')$.

By induction we define a sequence $M_1, \ldots, M_{2n+1}$ of subsets of S' as follows: Let M_{2n+1} be a T_{2n+1}-minimal subset of $\delta'(s', T_{2n+1})$.

If M_i has been defined, define M_{i-1} as a T_{i-1}-minimal subset of $\delta'(u_i, T_{i-1})$, for some $u_i \in M_i$. Since $T_{i-1} \subset T_i$, $M_{i-1} \subseteq M_i$.

Let $t \in M_1$. We construct a sequence $x_1, \ldots, x_{2n+1}$ of elements of Σ^* as follows: Let $t_i = \delta'(t, y_i) \in M_i$. Since M_i is T_i-minimal, for some $z_i \in T_i$, $\delta'(t_i, z_i) = t$. Put $x_1 = y_1 z_1$, $x_{i+1} = x_i y_{i+1} z_{i+1}$ $(i < 2n+1)$. We claim that x_i is an s-realization of N_i: To prove this assertion we proceed by induction on i:

Basis. $i = 1$. $z_i = y_i^m$ for some m, $x_1 = y_1^{m+1}$. y_1 is an s-realization of N_1; therefore, $x_1 = y_1^{m+1}$ is an s-realization of N_1.

Inductive step. Suppose that x_i is an s-realization of N_i, y_{i+1} is an s-realization of N_{i+1}, and z_{i+1} is the concatenation of y_j''s $(j \leq i+1)$, each of y_j's is an s-realization of N_j. Since $N_j \subseteq N_{i+1}$ for $1 \leq j \leq i+1$, x_{i+1} is an s-realization of N_{i+1}.

By the construction above, $\delta'(t, x_i) = t$ $(i = 1, \ldots, 2n+1)$.

Let N'_i denote the cycle of S' realizable by x_i. It follows that $N'_1 \subset N'_2 \subset \cdots \subset N'_{2n+1}$, because $x_{i+1} = x_i y_{i+1} z_{i+1}$.

Since $t \in M_1 \subseteq \cdots \subseteq M_{2n-1} \subseteq \delta'(s', T_{2n-1})$, it follows that, for some $w \in T_1$, $t = \delta'(s', w)$. Hence, $\delta(s, w) = s$. For each $i = 1, \ldots, 2n+1$ consider $\alpha_i = xwx_i^\omega$. $\mathrm{In}(\boldsymbol{A}(\alpha_i)) = N_i$, $\mathrm{In}(\boldsymbol{A}'(\alpha_i)) = N'_i$.

Since $T(\boldsymbol{A}) = T(\boldsymbol{A}')$, $N_i \in \boldsymbol{F}$ if and only if $N'_i \in \boldsymbol{F}'$. Denoting N'_{2i-1} by F'_i $(i = 1, \ldots, n+1)$ and N'_{2j} by D'_j $(j = 1, \ldots, n)$, we obtain that $F'_1 \subset D'_1 \subset \cdots \subset D'_n \subset F'_{n+1}$, and $F'_i \in \boldsymbol{F}'$ for $i = 1, \ldots, n+1$, $D'_j \notin \boldsymbol{F}'$ for $j = 1, \ldots, n$. □

4. Connection between hierarchy classes

The following result is an easy corollary of Theorem 2.12.

4.1. Theorem. *We have the following relations*:

$$\begin{tikzcd}[column sep=small]
\mathbf{LB}_{n-1} \subset \mathbf{LRB}_n & & \mathbf{LB}_n \\
& \mathbf{LB}_n \cap \mathbf{RB}_n \arrow[ul, phantom, sloped, "\supset"] \arrow[ur, phantom, sloped, "\subset"] & \\
\mathbf{RB}_{n-1} \subset \boldsymbol{B}_{n-1} \arrow[ur, phantom, sloped, "\subset"] & & \mathbf{RB}_n \subset \boldsymbol{B}_n. \arrow[ul, phantom, sloped, "\supset"]
\end{tikzcd}$$

Let A and A' be ω-regular languages. Given the positions of A and A' in the Boolean hierarchy we can determine the position of $\bar{A}$, $A \cup A'$, and $A \cap A'$.

4.2. Theorem. *We have*:

(a)

A	$\bar{A}$
$\boldsymbol{B}_n$	$\mathbf{LRB}_{n+1}$
$\mathbf{LB}_n$	$\mathbf{RB}_n$
$\mathbf{RB}_n$	$\mathbf{LB}_n$
$\mathbf{LRB}_n$	$\boldsymbol{B}_{n-1}$

The left and the right entry of each row denote the positions of A and $\bar{A}$ respectively.

(b)

$A \cup A'$	$\boldsymbol{B}_m$	$\mathbf{LB}_m$	$\mathbf{RB}_m$	$\mathbf{LRB}_m$
$\boldsymbol{B}_n$	$\boldsymbol{B}_{m+n}$	$\mathbf{LRB}_{m+n}$	$\mathbf{RB}_{m+n}$	$\mathbf{LRB}_{m+n}$
$\mathbf{LB}_n$	$\mathbf{LB}_{m+n}$	$\mathbf{LB}_{m+n-1}$	$\mathbf{LRB}_{m+n}$	$\mathbf{LRB}_{m+n-1}$
$\mathbf{RB}_n$	$\mathbf{RB}_{n+n}$	$\mathbf{LRB}_{m+n}$	$\mathbf{RB}_{m+n-1}$	$\mathbf{LRB}_{m+n-1}$
$\mathbf{LRB}_n$	$\mathbf{LRB}_{m+n}$	$\mathbf{LRB}_{m+n-1}$	$\mathbf{LRB}_{m+n-1}$	$\mathbf{LRB}_{m+n-2}$

The position of A is given by the corresponding entry of the left column, the position of A' is given by the corresponding entry of the top row, and an upper bound on the position of $A \cup A'$ is the entry of the intersection of the corresponding column and row.

(c)

$A \cap A'$	$\boldsymbol{B}_m$	$\mathbf{LB}_m$	$\mathbf{RB}_m$	$\mathbf{LRB}_m$
$\boldsymbol{B}_n$	$\boldsymbol{B}_{m+n-1}$	$\boldsymbol{B}_{m+n-1}$	$\boldsymbol{B}_{m+n-1}$	$\boldsymbol{B}_{m+n-1}$
$\mathbf{LB}_n$	$\boldsymbol{B}_{m+n-1}$	$\mathbf{LB}_{m+n-1}$	$\boldsymbol{B}_{m+n-1}$	$\mathbf{LB}_{m+n-1}$
$\mathbf{RB}_n$	$\boldsymbol{B}_{m+n-1}$	$\boldsymbol{B}_{m+n-1}$	$\mathbf{RB}_{m+n-1}$	$\mathbf{RB}_{m+n-1}$
$\mathbf{LRB}_n$	$\boldsymbol{B}_{m+n-1}$	$\mathbf{LB}_{m+n-1}$	$\mathbf{RB}_{m+n-1}$	$\mathbf{LRB}_{m+n-1}$

The position of A is given by the corresponding entry of the left column, the position of A' is given by the corresponding entry of the top row, and an upper bound on the position of $A \cap A'$ is the entry of the intersection of the corresponding column and row. (*The bound is quite surprising*: *an obvious one is 'about' $\boldsymbol{B}_{mn}$*.)

Proof

(a) Follows from Theorem 2.12 and Corollary 3.9.

(b) Follows from Lemma 2.5.

To prove (c) we apply (a) and (b) to $\overline{\bar{A} \cup \bar{A}'}$ $(= A \cap A')$. $\square$

The following theorem gives another description of the hierarchy classes.

4.3. Theorem. *Let $P = P(A_1, \ldots, A_m)$ be a Boolean expression over $\boldsymbol{B}$, and let P_u, P_i, and P_c denote the number of occurrences of $\cup$, $\cap$ and $-$ in P, respectively.*

(a) *Suppose that $P \in \boldsymbol{B}_n/(\mathbf{LB}_n \cup \mathbf{RB}_n)$. Then*:

(i) $P_c \geqslant n$.

(ii) $P_u + P_i \geqslant 2n - 1$.

(iii) *If $P_c = n$, and $P_u + P_i = 2n - 1$, then*

$$P = \bigcup_{i=1}^{n} (A_{k_{2i-1}} \cap \bar{A}_{k_{2i}}), \quad \text{where } 1 \leqslant k_j \leqslant m \ \ (j = 1, \ldots, 2n).$$

(b) *Suppose that $P \in \mathbf{LB}_n/\mathbf{LRB}_n$. Then*:

(i) $P_c \geqslant n$.

(ii) $P_u + P_i \geqslant 2n - 2$.

(iii) *If $P_c = n$, and $P_u + P_i = 2n - 2$, then*

$$P = \bar{A}_{k_2} \cup \bigcup_{i=2}^{n} (A_{k_{2i-1}} \cap \bar{A}_{k_{2i}}), \quad \text{where } 1 \leqslant k_j \leqslant m \ \ (j = 2, \ldots, 2n).$$

(c) *Suppose that $P \in \mathbf{RB}_n/\mathbf{LRB}_n$. Then*:

(i) $P_c \geqslant n - 1$.

(ii) $P_u + P_i \geqslant 2n - 2$.

(iii) *If $P_c = n - 1$, and $P_u + P_i = 2n - 2$, then*

$$P = \bigcup_{i=1}^{n-1} (A_{k_{2i-1}} \cap \bar{A}_{k_{2i}}) \cup A_{k_{2n-1}}, \quad \text{where } 1 \leqslant k_j \leqslant m \ \ (j = 1, \ldots, 2n - 1).$$

(d) *Suppose that $P \in \mathbf{LRB}_n/\boldsymbol{B}_{n-1}$. Then*:

(i) $P_c \geqslant n - 1$.

(ii) $P_u + P_i \geqslant 2n - 3$.

(iii) *If $P_c = n - 1$, and $P_u + P_i = 2n - 3$, then*

$$P = \bar{A}_{k_2} \cup \bigcup_{i=2}^{n-1} (A_{k_{2i-1}} \cap \bar{A}_{k_{2i}}) \cup A_{k_{2n-1}}, \quad \text{where } 1 \leqslant k_j \leqslant m \ \ (j = 2, \ldots, 2n - 1).$$

Proof. Part (i) of (a)–(d): Let P be an expression of the minimal complexity which violates (i) of one of (a)–(d). Suppose that P violates (i) of (a). If $P = \bar{P}'$, then, by Theorem 4.2(a), $P' \in \mathbf{LRB}_{n+1}/\boldsymbol{B}_n$. Since the theorem holds for P', $P'_c \geqslant n$. Therefore, $P_c = P'_c + 1 \geqslant n + 1$. This contradicts our assumption. Cases of $P = P' \cup P''$ and $P = P' \cap P''$ can be treated in the same manner, using points (b) and (c) of Theorem 4.2.

Points (ii) and (iii) of (a)–(d) can be proved in a similar fashion. $\square$

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